Complete Constant Mean Curvature surfaces in homogeneous spaces

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Abstract

In this paper we classify complete surfaces of constant mean curvature whose Gaussian curvature does not change sign in a simply connected homogeneous manifold with a 4-dimensional isometry group.

1 Introduction

In 1966, T. Klotz and R. Ossermann showed the following:

Theorem [KO]: *A complete $H-$surface in $\mathbb{R}^3$ whose Gaussian curvature K does not change sign is either a sphere, a minimal surface, or a right circular cylinder.*

The above result was extended to $\mathbb{S}^3$ by D. Hoffman [H], and to $\mathbb{H}^3$ by R. Tribuzy [T] with an extra hypothesis if K is non-positive. The additional hypothesis says that, when $K \leq 0$, one has $H^2 - K - 1 > 0$.

In recent years, the study of $H-$surfaces in product spaces and, more generally, in a homogeneous three-manifold with a 4-dimensional isometry group is quite active (see [AR, AR2], [CoR], [ER], [FM, FM2], [DH] and references therein).

The aim of this paper is to extend the above Theorem to homogeneous spaces with a 4-dimensional isometry group. These homogeneous space are denoted by $\mathbb{E}(\kappa, \tau)$, where κ and τ are constant and $\kappa - 4\tau^2 \neq 0$. They can be classified as: the product spaces $\mathbb{H}^2 \times \mathbb{R}$ if $\kappa = -1$

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and $\tau = 0$, or $\mathbb{S}^2 \times \mathbb{R}$ if $\kappa = 1$ and $\tau = 0$, the Heisenberg space Nil_3 if $\kappa = 0$ and $\tau = 1/2$, the Berger spheres $\mathbb{S}^3_{Berger}$ if $\kappa = 1$ and $\tau \neq 0$, and the universal covering of $\mathrm{PSL}(2, \mathbb{R})$ if $\kappa = -1$ and $\tau \neq 0$.

The paper is organized as follows. In Section 2, we establish the definitions and necessary equations for an $H-$surface. We also state here two classification results for $H-$surfaces. We prove them in Section 5 and Section 6 for the sake of completeness.

Section 3 is devoted to the classification of $H-$surfaces with non-negative Gaussian curvature,

Theorem 3.1. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H-$surface with $K \geq 0$. Then, Σ is either a rotational sphere (in particular, $4H^2 + \kappa > 0$), or a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

In Section 4 we continue with the classification of $H-$surfaces with non-positive Gaussian curvature.

Theorem 4.1. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H-$surface with $K \leq 0$ and $H^2 + \tau^2 - |\kappa - 4\tau^2| > 0$. Then, Σ is a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

In the Appendix, we give a result, which we think is of independent interest, concerning differential operators on a Riemannian surface Σ of the form $\Delta + g$, acting on $C^2(\Sigma)-$functions, where Δ is the Laplacian with respect to the Riemannian metric on Σ and $g \in C^0(\Sigma)$.

2 The geometry of surfaces in homogeneous spaces

Henceforth $\mathbb{E}(\kappa, \tau)$ denotes a complete simply connected homogeneous three-manifold with $4-$dimensional isometry group. Such a three-manifold can be classified in terms of a pair of real numbers (κ, τ) satisfying $\kappa - 4\tau^2 \neq 0$. In fact, these manifolds are Riemannian submersions over a complete simply-connected surface $\mathbb{M}^2(\kappa)$ of constant curvature κ, $\pi : \mathbb{E}(\kappa, \tau) \longrightarrow \mathbb{M}^2(\kappa)$, and translations along the fibers are isometries, therefore they generate a Killing field ξ, called the *vertical field*. Moreover, τ is the real number such that $\overline{\nabla}_X \xi = \tau X \wedge \xi$ for all vector fields X on the manifold. Here, $\overline{\nabla}$ is the Levi-Civita connection of the manifold and $\wedge$ is the cross product.

Let Σ be a complete $H-$surface immersed in $\mathbb{E}(\kappa, \tau)$. By passing to a $2-$sheeted covering space of Σ, we can assume Σ is orientable. Let N be a unit normal to Σ. In terms of a conformal parameter z of Σ, the first, $\langle \cdot, \cdot \rangle$, and second, II, fundamental forms are given by

$$\begin{aligned} \langle \cdot, \cdot \rangle &= \lambda \, |dz|^2 \\ II &= p \, dz^2 + \lambda \, H \, |dz|^2 + \overline{p} \, d\bar{z}^2, \end{aligned} \tag{2.1}$$

where $p \, dz^2 = \langle -\nabla_{\partial_z} N, \partial_z \rangle \, dz^2$ is the Hopf differential of Σ.

Set $\nu = \langle N, \xi \rangle$ and $T = \xi - \nu N$, i.e., ν is the normal component of the vertical field ξ, called the *angle function*, and T is the tangent component of the vertical field.

First we state the following necessary equations on Σ which were obtained in [FM].

Lemma 2.1. *Given an immersed surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$, the following equations are satisfied:*

$$
\begin{aligned}
K &= K_e + \tau^2 + (\kappa - 4\tau^2)\,\nu^2 && (2.2)\\
p_{\bar z} &= \frac{\lambda}{2}\,(H_z + (\kappa - 4\tau^2)\,\nu\,A) && (2.3)\\
A_{\bar z} &= \frac{\lambda}{2}\,(H + i\tau)\,\nu && (2.4)\\
\nu_z &= -(H - i\tau)\,A - \frac{2}{\lambda}\,p\,\overline{A} && (2.5)\\
|A|^2 &= \frac{1}{4}\,\lambda\,(1 - \nu^2) && (2.6)\\
A_z &= \frac{\lambda_z}{\lambda}\,A + p\,\nu && (2.7)
\end{aligned}
$$

where $A = \langle \xi, \partial_z \rangle$, K_e the extrinsic curvature and K the Gauss curvature of Σ.

For an immersed $H-$surface $\Sigma \subset \mathbb{E}(\kappa, \tau)$ there is a globally defined quadratic differential, called the *Abresch-Rosenberg* differential, which in these coordinates is given by (see [AR2]):

$$Q\,dz^2 = (2(H + i\tau)\,p - (\kappa - 4\tau^2)A^2)\,dz^2,$$

following the notation above.

It is not hard to verify this quadratic differential is holomorphic on an $H-$surface using (2.3) and (2.4),

Theorem 2.1 ([AR],[AR2])**.** *$Q\,dz^2$ is a holomorphic quadratic differential on any $H-$surface in $\mathbb{E}(\kappa, \tau)$.*

Associated to the Abresch-Rosenberg differential we define the smooth function $q : \Sigma \longrightarrow [0, +\infty)$ given by

$$q = \frac{4|Q|^2}{\lambda^2}.$$

By means of Theorem 2.1, q either has isolated zeroes or vanishes identically. Note that q does not depend on the conformal parameter z, hence q is globally defined on Σ.

We continue this Section establishing some formulae relating the angle function, q and the Gaussian curvature.

Lemma 2.2. *Let Σ be an $H-$surface immersed in $\mathbb{E}(\kappa,\tau)$. Then the following equations are satisfied:*

$$\|\nabla\nu\|^2 = \frac{4H^2+\kappa-(\kappa-4\tau^2)\nu^2}{4(\kappa-4\tau^2)}\left(4(H^2-K_e)+(\kappa-4\tau^2)(1-\nu^2)\right) - \frac{q}{\kappa-4\tau^2} \tag{2.8}$$

$$\Delta\nu = -\left(4H^2+2\tau^2+(\kappa-4\tau^2)(1-\nu^2)-2K_e\right)\nu. \tag{2.9}$$

Moreover, away from the isolated zeroes of q, we have

$$\Delta \ln q = 4K. \tag{2.10}$$

Proof. From (2.5)

$$|\nu_z|^2 \;=\; \frac{4\,|p|^2\,|A|^2}{\lambda^2} + (H^2+\tau^2)\,|A|^2 + \frac{2\,(H+i\tau)}{\lambda}p\overline{A}^2 + \frac{2\,(H-i\tau)}{\lambda}\overline{p}A^2,$$

and taking into account that

$$|Q|^2 \;=\; 4\,(H^2+\tau^2)\,|p|^2 + (\kappa-4\tau^2)^2|A|^4 - (\kappa-4\tau^2)\left(2\,(H+i\tau)p\overline{A}^2 + 2\,(H-i\tau)\overline{p}A^2\right),$$

we obtain, using also (2.6), that

$$\begin{aligned}|\nu_z|^2 &= (H^2+\tau^2)|A|^2 + (H^2-K_e)|A|^2 + (\kappa-4\tau^2)\frac{|A|^4}{\lambda}\\ &\quad + 4\left(\frac{H^2+\tau^2}{\kappa-4\tau^2}\right)\frac{|p|^2}{\lambda} - \frac{|Q|^2}{(\kappa-4\tau^2)\lambda}\end{aligned}$$

where we have used that $4|p|^2 = \lambda^2(H^2-K_e)$ and $\kappa-4\tau^2\neq 0$. Thus

$$\begin{aligned}\|\nabla\nu\|^2 = \frac{4}{\lambda}|\nu_z|^2 &= (2H^2-K_e+\tau^2)(1-\nu^2) + \frac{\kappa-4\tau^2}{4}(1-\nu^2)^2\\ &\quad + 4\left(\frac{H^2+\tau^2}{\kappa-4\tau^2}\right)(H^2-K_e) - \frac{q}{\kappa-4\tau^2},\end{aligned}$$

finally, re-ordering in terms of H^2-K_e we have the expression.

On the other hand, by differentiating (2.5) with respect to $\bar{z}$ and using (2.7), (2.4) and (2.3), one gets

$$\nu_{z\bar{z}} \;=-(\kappa-4\tau^2)\,\nu\,|A|^2 - \frac{2}{\lambda}\,|p|^2\,\nu - \frac{H^2+\tau^2}{2}\,\lambda\,\nu.$$

Then, from (2.6),

$$\nu_{z\bar{z}} = -\frac{\lambda\,\nu}{4}\left((\kappa-4\tau^2)(1-\nu^2) + \frac{8\,|p|^2}{\lambda^2} + 2\,(H^2+\tau^2)\right)$$

thus

$$\Delta\nu = \frac{4}{\lambda}\nu_{z\bar{z}} = -\left((\kappa - 4\tau^2)(1-\nu^2) + 2(H^2 - K_e) + 2\left(H^2+\tau^2\right)\right)\nu.$$

Finally,

$$\Delta \ln q = \Delta \ln \frac{4|Q|^2}{\lambda^2} = -2\Delta \ln \lambda = 4K,$$

where we have used that $Q\,dz^2$ is holomorphic and the expression of the Gaussian curvature in terms of a conformal parameter. $\square$

Remark 2.1. *Note that* (2.9) *is nothing but the Jacobi equation for the Jacobi field* ν.

Next, we recall a definition in these homogeneous spaces.

Definition 2.1. *We say that* $\Sigma \subset \mathbb{E}(\kappa,\tau)$ *is a vertical cylinder over* α *if* $\Sigma = \pi^{-1}(\alpha)$, *where* α *is a curve on* $\mathbb{M}^2(\kappa)$.

It is not hard to verify that if α is a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$, then $\Sigma = \pi^{-1}(\alpha)$ is complete and has constant mean curvature H. Moreover, these cylinders are characterized by $\nu \equiv 0$.

We now state two results about the classification of $H-$surfaces. They will be used in Sections 3 and 4, but we prove them in Section 5 and Section 6 for the sake of clarity. The first one concerns $H-$surfaces for which the angle function is constant.

Theorem 2.2. *Let* $\Sigma \subset \mathbb{E}(\kappa,\tau)$ *be a complete* $H-$*surface with constant angle function. Then* Σ *is either a vertical cylinder over a complete curve of curvature* $2H$ *on* $\mathbb{M}^2(\kappa)$, *or a slice in* $\mathbb{H}^2\times\mathbb{R}$ *or* $\mathbb{S}^2\times\mathbb{R}$.

Remark 2.2. *Theorem 2.2 improves [ER, Lemma 2.3] for surfaces in* $\mathbb{H}^2\times\mathbb{R}$.

Of special interest for us are those $H-$surfaces for which the Abresch-Rosenberg differential is constant.

Theorem 2.3. *Let* $\Sigma \subset \mathbb{E}(\kappa,\tau)$ *be a complete* $H-$*surface with* q *constant.*

- *If* $q = 0$ *on* Σ, *then:*
  - *If* $H = 0 = \tau$, Σ *is a slice in* $\mathbb{H}^2\times\mathbb{R}$ *or* $\mathbb{S}^2\times\mathbb{R}$.
  - *If* $4H^2+\kappa > 0$, Σ *is a rotational embedded sphere* S_H *which also implies that* $K > 0$.

– *If $4H^2+\kappa=0$ and $\nu\equiv 0$ on Σ, Σ is a vertical cylinder over a complete curve of curvature $|\kappa|$. That is, Σ is either a vertical cylinder over a straight line in Nil_3, or a vertical cylinder over a horocycle in $\mathbb{H}^2\times\mathbb{R}$ or $\widetilde{\mathrm{PLS}(2,\mathbb{C})}$. Moreover, all these examples are flat.*

– *If $4H^2+\kappa\leq 0$ and ν is not constant, then Σ has a point with negative Gauss curvature.*

- *If $q\neq 0$ on Σ, then Σ is a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

3 Complete $H-$surfaces Σ with $K\geq 0$

Here we prove

Theorem 3.1. *Let $\Sigma\subset\mathbb{E}(\kappa,\tau)$ be a complete $H-$surface with $K\geq 0$. Then, Σ is either a rotational sphere (in particular, $4H^2+\kappa>0$), or a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

Proof. The proof goes as follows: First, we prove that Σ is a topological sphere or a complete non-compact parabolic surface. We show that when the surface is a topological sphere then it is a rotational sphere. If Σ is a complete non-compact parabolic surface, we prove that it is a vertical cylinder by means of Theorem 2.3.

Since $K\geq 0$ and Σ is complete, [KO, Lemma 5] implies that Σ is either a sphere or non-compact and parabolic.

If Σ is a sphere, then it is a rotational example (see [AR2] or [AR]). Thus, we can assume that Σ is non-compact and parabolic.

We can assume that q does not vanish identically in Σ. If q does vanish, then Σ is either a vertical cylinder over a straight line in Nil_3 or a vertical cylinder over a horocycle in $\mathbb{H}^2\times\mathbb{R}$ or $\widetilde{\mathrm{PLS}(2,\mathbb{C})}$. Note that we have used here that $K\geq 0$ and Theorem 2.3.

On the one hand, from the Gauss equation (2.2)

$$0\leq K=K_e+\tau^2+(\kappa-4\tau^2)\nu^2\leq K_e+\tau^2+|\kappa-4\tau^2|,$$

then

$$H^2-K_e\leq H^2+\tau^2+|\kappa-4\tau^2|. \tag{3.1}$$

On the other hand, using the very definition of $Q\,dz^2$, (3.1) and the inequality $|\xi_1+\xi_2|^2\leq$

$2(|\xi_1|^2 + |\xi|^2)$ for $\xi_1, \xi_2 \in \mathbb{C}$, we obtain

$$\begin{aligned}
\frac{q}{2} = \frac{2|Q|^2}{\lambda^2} &\leq 4(H^2+\tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa-4\tau^2)^2\frac{4|A|^4}{\lambda^2} \\
&= 4(H^2+\tau^2)(H^2-K_e) + \frac{(\kappa-4\tau^2)^2}{4}(1-\nu^2)^2 \\
&\leq 4(H^2+\tau^2)(H^2-K_e) + \frac{(\kappa-4\tau^2)^2}{4} \\
&\leq 4(H^2+\tau^2)(H^2+\tau^2+|\kappa-4\tau^2|) + \frac{(\kappa-4\tau^2)^2}{4}.
\end{aligned}$$

So, from (2.10), $\Delta \ln q = 4K \geq 0$ and $\ln q$ is a bounded subharmonic function on a non-compact parabolic surface Σ and since the value $-\infty$ is allowed at isolated points (see [AS]), q is a positive constant (recall that we are assuming that q does not vanishes identically). Therefore, Theorem 2.3 gives the result.

$\square$

4 Complete $H-$surfaces Σ with $K \leq 0$

Theorem 4.1. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete $H-$surface with $K \leq 0$ and $H^2+\tau^2-|\kappa-4\tau^2| > 0$. Then, Σ is a complete vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

Proof. We divide the proof in two cases, $\kappa-4\tau^2<0$ and $\kappa-4\tau^2>0$.

Case $\kappa-4\tau^2<0$:

On the one hand, since $K\leq 0$, we have

$$H^2-K_e \geq H^2+\tau^2+(\kappa-4\tau^2)\nu^2 \geq H^2+\kappa-3\tau^2,$$

from the Gauss Equation (2.2). Therefore, from (2.8) and $\kappa-4\tau^2<0$, we obtain:

$$\begin{aligned}
q &\geq 4(H^2+\tau^2)(H^2-K_e) + (\kappa-4\tau^2)(1-\nu^2)\left(H^2+\tau^2+H^2-K_e+\frac{\kappa-4\tau^2}{4}(1-\nu^2)\right) \\
&= (H^2-K_e)\left(4H^2+4\tau^2+(\kappa-4\tau^2)(1-\nu^2)\right) \\
&\qquad + (H^2+\tau^2)(\kappa-4\tau^2)(1-\nu^2) + \frac{(\kappa-4\tau^2)^2}{4}(1-\nu^2)^2 \\
&\geq \left(H^2+\tau^2+(\kappa-4\tau^2)\nu^2\right)\left(4H^2+4\tau^2+(\kappa-4\tau^2)(1-\nu^2)\right) \\
&\qquad + (H^2+\tau^2)(\kappa-4\tau^2)(1-\nu^2) + \frac{(\kappa-4\tau^2)^2}{4}(1-\nu^2)^2,
\end{aligned}$$

note that the last inequality holds since $4H^2 + 4\tau^2 + (\kappa - \tau^2)(1 - \nu^2) \geq 4H^2 + \kappa > 0$. $4H^2 + \kappa > 0$ follows from

$$0 < 4(H^2 + \tau^2) - |\kappa - 4\tau^2| = 4H^2 + \kappa.$$

Set $a := H^2 + \tau^2$ and $b := \kappa - 4\tau^2$. Define the real smooth function $f : [-1, 1] \longrightarrow \mathbb{R}$ as

$$f(x) = (a + bx^2)(4a + b(1 - x^2)) + ab(1 - x^2) + \frac{b^2}{4}(1 - x^2)^2. \tag{4.1}$$

Note that $q \geq f(\nu)$ on Σ, $f(\nu)$ is just the last part in the above inequality involving q. It is easy to verify that the only critical point of f in $(-1, 1)$ is $x = 0$. Moreover,

$$f(0) = (4a + b)^2/4 > 0 \quad \text{and} \quad f(\pm 1) = 4a(a + b) > 0.$$

Actually, $f : \mathbb{R} \longrightarrow \mathbb{R}$ has two others critical points, $x = \pm\sqrt{\frac{4a+b}{3|b|}}$, but here, we have used that

$$\frac{4a + b}{3|b|} > 1,$$

since $0 < 4(H^2 + \kappa - 3\tau^2) = (4H^2 + \kappa) - 3|\kappa - 4\tau^2| = (4a + b) - 3|b|$.

So, set $c = \min\{f(0), f(\pm 1)\} > 0$, then

$$q \geq f(\nu) \geq c > 0.$$

Now, from (2.10) and $q \geq c > 0$ on Σ, it follows that $ds^2 = \sqrt{q}I$ is a complete flat metric on Σ and

$$\Delta^{ds^2} \ln q = \frac{1}{\sqrt{q}} \Delta \ln q = \frac{4K}{\sqrt{q}} \leq 0.$$

Since q is bounded below by a positive constant and (Σ, ds^2) is parabolic, then $\ln q$ is constant which implies that q is a positive constant (recall q is bounded below by a positive constant). Thus, the result follows from Theorem 2.3. The case $\kappa - 4\tau^2 < 0$ is proved.

Case $\kappa - 4\tau^2 > 0$:

Set $w_1 := 2(H + i\tau)\dfrac{p}{\lambda}$ and $w_2 := (\kappa - 4\tau^2)\dfrac{A^2}{\lambda}$, i.e., $q = 4|w_1 + w_2|^2$. Then

$$|w_1|^2 = (H^2 + \tau^2)(H^2 - K_e) \geq (H^2 + \tau^2)^2$$

$$|w_2|^2 = \frac{(\kappa - 4\tau^2)^2}{16}(1 - \nu^2)^2 \leq \left(\frac{\kappa - 4\tau^2}{4}\right)^2,$$

where we have used that $H^2 - K_e \geq H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 \geq H^2 + \tau^2$, since $K \leq 0$ and $\kappa - 4\tau^2 > 0$.

Let us recall a well known inequality for complex numbers, let $\xi_1, \xi_2 \in \mathbb{C}$ then $|\xi_1 + \xi_2|^2 \geq ||\xi_1| - |\xi_2||^2$. Thus,

$$\begin{aligned} \frac{1}{4}q \geq ||w_1| - |w_2||^2 &\geq \left|(H^2+\tau^2) - \frac{|\kappa - 4\tau^2|}{4}\right|^2 \\ &= \frac{1}{16}\left|4(H^2+\tau^2) - |\kappa - 4\tau^2|\right|^2 > 0. \end{aligned}$$

So, q is bounded below by a positive constant then, arguing as in the previous case, q is constant. Thus, the result follows from Theorem 2.3. The case $\kappa - 4\tau^2 > 0$ is proved.

□

Remark 4.1. *Note that in the above Theorem, in the case $\kappa - 4\tau^2 > 0$, we only need to assume $4(H^2+\tau^2) - |\kappa - 4\tau^2| > 0$.*

5 Complete $H-$surfaces with constant angle function

We classify here the complete $H-$surfaces in $\mathbb{E}(\kappa, \tau)$ with constant angle function. The purpose is to take advantage of this classification result in the next Section.

Theorem 2.2. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be a complete $H-$surface with constant angle function. Then Σ is either a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$, or a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. We can assume that $\nu \leq 0$. We will divide the proof in three cases:

- $\nu = 0$: In this case, Σ must be a vertical cylinder over a complete curve of geodesic curvature $2H$ on $\mathbb{M}^2(\kappa)$.
- $\nu = -1$: From (2.4), $\tau = 0$ and $H = 0$, then Σ is a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.
- $-1 < \nu < 0$: We prove that this case is impossible. From (2.5), we have

$$(H - i\tau)A = -\frac{2p}{\lambda}\overline{A} \tag{5.1}$$

then

$$H^2 + \tau^2 = \frac{4|p|^2}{\lambda^2} = H^2 - K_e$$

since $|A|^2 \neq 0$ from (2.6), so $K_e = -\tau^2$ on Σ.

Thus, from (2.9), we have

$$4H^2 + 4\tau^2 + (\kappa - 4\tau^2)(1 - \nu^2) = 0. \tag{5.2}$$

Now, using the definition of q, (5.1), (5.2) and $K_e = -\tau^2$, we have

$$\begin{aligned} q = \frac{4|Q|^2}{\lambda^2} &= 4(H^2+\tau^2)\frac{4|p|^2}{\lambda^2} + (\kappa - 4\tau^2)^2 \frac{4|A|^4}{\lambda^2} \\ &\quad - 4\frac{\kappa - 4\tau^2}{\lambda^2}\left(2\,(H+i\tau)p\overline{A}^2 + 2\,(H - i\tau)\overline{p}A^2\right) \\ &= 4(H^2+\tau^2)(H - K_e) + (\kappa - 4\tau^2)\frac{(1-\nu^2)^2}{4} + 2(\kappa - 4\tau^2)(1-\nu^2)(H^2+\tau^2) \\ &= \frac{1}{4}\left(4H^2 + (\kappa - 4\tau^2)(1-\nu^2) + 4\tau^2\right)^2 = 0 \end{aligned}$$

that is, q vanishes identically on Σ. This contradicts Theorem 2.3 since $0 < \nu^2 < 1$ is constant.

$\square$

6 Complete $H-$surfaces with q constant

Here, we prove the classification result for complete $H-$surfaces in $\mathbb{E}(\kappa,\tau)$ employed in the proof of Theorem 3.1 and Theorem 4.1.

Theorem 2.3. *Let $\Sigma \subset \mathbb{E}(\kappa,\tau)$ be a complete $H-$surface with q constant.*

- *If $q = 0$ on Σ, then:*
  - *If $H = 0 = \tau$, Σ is a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.*
  - *If $4H^2 + \kappa > 0$, Σ is a rotational embedded sphere S_H which also implies that $K > 0$.*
  - *If $4H^2 + \kappa = 0$ and $\nu \equiv 0$ on Σ, Σ is a vertical cylinder over a complete curve of curvature $|\kappa|$. That is, Σ is either a vertical cylinder over a straight line in Nil_3, or a vertical cylinder over a horocycle in $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{PLS}}(2,\mathbb{C})$. Moreover, all these examples are flat.*
  - *If $4H^2 + \kappa \leq 0$ and ν is not constant, then Σ has a point with negative Gauss curvature.*
- *If $q \neq 0$ on Σ, then Σ is a vertical cylinder over a complete curve of curvature $2H$ on $\mathbb{M}^2(\kappa)$.*

The case $q = 0$ has been treated extensively when the target manifold is a product space, but is has not been established explicitly when $\tau \neq 0$. So, we assemble the results in [AR], [AR2] for the readers convenience.

Lemma 6.1. *Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$ be an $H-$surface whose Abresch-Rosenberg differential vanishes. Then Σ is either a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ if $H = 0 = \tau$, or Σ is invariant by a one-parameter group of isometries of $\mathbb{E}(\kappa, \tau)$.*

Moreover, the Gauss curvature of these examples is

- *If $4H^2 + \kappa > 0$, then $K > 0$ they are the rotationally invariant spheres.*
- *If $4H^2 + \kappa = 0$ and $\nu \equiv 0$, then $K \equiv 0$ and Σ is either a vertical plane in Nil_3, or a vertical cylinder over a horocycle in $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{PSL}(2, \mathbb{C})}$.*
- *There exists a point with negative Gauss curvature in the remaining cases.*

Proof. The idea of the proof for product spaces that we use below, can be found in [dCF] and [FM].

If $H = 0 = \tau$, from the definition of the Abresch-Rosenberg differential, we have

$$0 = -(\kappa - 4\tau)A^2,$$

that is, $\nu^2 = \pm 1$ using (2.6). Thus, Σ is a slice in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$.

If $H \neq 0$ or $\tau \neq 0$, we have

$$2(H + i\tau)p = (\kappa - 4\tau^2)A^2,$$

from where we obtain, taking modulus,

$$H^2 - K_e = \frac{(\kappa - 4\tau^2)^2(1 - \nu^2)^2}{16(H^2 + \tau^2)} \tag{6.1}$$

Replacing the above equation in (2.5),

$$(H + i\tau)\nu_z = -\frac{1}{4}(4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2)A,$$

and taking modulus,

$$|\nu_z|^2 = g(\nu)^2|A|^2, \;\; g(\nu) = \frac{4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2}{4\sqrt{H^2 + \tau^2}}. \tag{6.2}$$

Assume that ν is not constant. Let $p \in \Sigma$ be a point where $\nu_z(p) \neq 0$ and let $\mathcal{U}$ be a neighborhood of that point p where $\nu_z \neq 0$ (we can assume $\nu^2 \neq 1$ at p). In particular, $g(\nu) \neq 0$ in $\mathcal{U}$ from (6.2). Now, replacing (6.2) in (2.6), we obtain

$$\lambda = \frac{4|\nu_z|^2}{(1 - \nu^2)g(\nu)^2}. \tag{6.3}$$

Thus, putting (6.1) and (6.3) in the Jacobi equation (2.9)

$$\nu_{z\bar{z}} = -2\frac{\nu|\nu_z|^2}{1-\nu^2}. \tag{6.4}$$

So, define the real function $s := \operatorname{arctgh}(\nu)$ on $\mathcal{U}$. Such a function is harmonic by means of (6.4), thus we can consider a new conformal parameter w for the first fundamental form so that $s = \operatorname{Re}(w)$, $w = s + it$.

Since $\nu = \operatorname{tgh}(s)$ by the definition of s, we have that $\nu \equiv \nu(s)$, i.e., it only depends on one parameter. Thus, we have $\lambda \equiv \lambda(s)$ and $T \equiv T(s)$ from (6.3) and (6.2) respectively, and $p \equiv p(s)$ by the definition of the Abresch-Rosenberg differential. That is, all the fundamental data of Σ depend only on s.

Thus, the surface is invariant by a one parameter group of isometries, the proof of this is the same as in [FM] for surfaces with vanishing Abresch-Rosenberg differential in product spaces.

Let us prove the claim about the Gauss curvature. Using the Gauss Equation (2.2) in (6.1), one gets

$$H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2 - K = \frac{(\kappa-4\tau^2)^2(1-\nu^2)^2}{16(H^2+\tau^2)}.$$

Set $a := 4(H^2+\tau^2)$ and $b := \kappa - 4\tau^2$, then one can check easily that the above equality can be expressed as

$$4aK = a^2 - b^2 + (2a+b)^2 - (2a + b(1-\nu^2))^2.$$

So, if $4H^2 + \kappa > 0$ then $a > |b|$ and $K > 0$, that is, Σ is a topological sphere since it is complete. If $4H^2 + \kappa = 0$, $a = -b$ and the equation reads as

$$4aK = a^2(1 - (1+\nu^2)^2),$$

that is, Σ has a point with negative Gauss curvature unless $\nu \equiv 0$.

If $4H^2 + \kappa < 0$, it is easy to check that there at least one point with negative curvature. $\square$

Proof of Theorem 2.3. We focus on the case $q \neq 0$ because Lemma 6.1 gives the classification when $q = 0$.

Suppose ν is not constant in Σ. Since $q = c^2 > 0$, we can consider a conformal parameter z so that $\langle \cdot, \cdot \rangle = |dz|^2$ and $Q\,dz^2 = c\,dz^2$ on Σ. Thus,

$$Q = c = 2(H + i\tau)p - (\kappa - 4\tau^2)A^2.$$

First, note that we can assume that $H \neq 0$ or $\tau \neq 0$, otherwise ν would be constant. So, from (2.5), we have

$$(H + i\tau)\nu_z = -(H^2 + \tau^2 + \frac{\kappa - 4\tau^2}{4}(1-\nu^2))A - c\overline{A},$$

where we have used $2(H+i\tau)p = c + (\kappa - 4\tau^2)A^2$. That is,

$$4(H^2+\tau^2)\left\|\nabla\nu\right\|^2 = \left(g(\nu)+4c\right)^2(1-\nu^2), \tag{6.5}$$

where

$$g(\nu) := 4H^2 + \kappa - (\kappa - 4\tau^2)\nu^2. \tag{6.6}$$

Combining (2.8) and (6.5), we get that ν is constant since q is constant. Therefore, by means of Theorem 2.2, Σ is a complete vertical cylinder.

From (2.10), Σ is flat and $H^2 - K_e = H^2 + \tau^2 + (\kappa - 4\tau^2)\nu^2$ by (2.2), joining this last equation to (2.8) we obtain using the definition of $g(\nu)$ given in (6.6)

$$\left\|\nabla\nu\right\|^2 = \frac{g(\nu)^2}{4(\kappa-4\tau^2)} + \nu^2 g(\nu) - \frac{c^2}{\kappa - 4\tau^2}. \tag{6.7}$$

Putting together (6.5) and (6.7) we obtain a polynomial expression in ν^2 with coefficients depending on $a := 4(H^2+\tau^2)$, $b := \kappa - 4\tau^2$ and c,

$$P(\nu^2) := C(a,b,c)\nu^6 + \text{ lower terms } = 0,$$

but one can easily check that the coefficient in ν^6 is $C(a,b,c) = -ab^2 \neq 0$, a contradiction. Thus ν is constant, and so, by means of Theorem 2.2, Σ is a vertical cylinder over a complete curve of curvature $2H$.

$\square$

7 Appendix

Let Σ be a connected Riemannian surface. We establish in this Appendix a result which we think is of independent interest, concerning differential operators of the form $\Delta + g$, acting on $C^2(\Sigma)-$functions, where Δ is the Laplacian with respect to the Riemannian metric on Σ and $g \in C^0(\Sigma)$.

Lemma 7.1. *Let $g \in C^0(\Sigma)$, $v \in C^2(\Sigma)$ such that $\left\|\nabla v\right\|^2 \leq h\, v^2$ on Σ, h is a non-negative continuos function on Σ, and $\Delta v + gv = 0$ in Σ. Then either v never vanishes or v vanishes identically on Σ.*

Proof. Set $\Omega = \{p \in \Sigma \,:\, v(p) = 0\}$. We will show that either $\Omega = \emptyset$ or $\Omega = \Sigma$.

So, let us assume that $\Omega \neq \emptyset$. If we prove that Ω is an open set then, since Ω is closed and Σ is connected, $\Omega = \Sigma$. Let $p \in \Omega$ and $\mathcal{B}(R) \subset \Sigma$ be the geodesic ball centered at p of radius R. Such a geodesic ball is relatively compact in Σ.

Set $\phi = v^2/2 \geq 0$. Then

$$\Delta\phi = v\Delta v + \left\|\nabla v\right\|^2 = -gv^2 + \left\|\nabla v\right\|^2 \leq -2(g-h)\phi,$$

that is,

$$-\Delta\phi - 2(g-h)\phi \geq 0. \tag{7.1}$$

Define $\beta := \min\{\inf_\Omega\{2(g-h)\}, 0\} \leq 0$. Then, $\psi = -\phi$ satisfies

$$\Delta\psi + \beta\psi = -\Delta\phi - \beta\phi \geq -\Delta\phi - 2(g-h)\phi \geq 0,$$

where we have used (7.1).

Since we are assuming that v has a zero at an interior point of $\mathcal{B}(R)$, $\beta \leq 0$ and ψ has a non-negative maximum at p, the Maximum Principle [GT, Theorem 3.5] implies that ψ is constant and so v is constant as well, i.e, $v \equiv 0$ in $\mathcal{B}(R)$. Then $\mathcal{B}(R) \subset \Omega$, and Ω is an open set. Thus $\Omega = \Sigma$. $\square$

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